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VISIBILITY PROBABILITIES AND

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IN THE PLANE FOR POISSON SHADOWING PROCESSES

By

Micha Yadin and S. Zacks
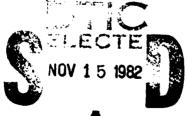
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VISIBILITY PROBABILITIES AND MOMENTS OF MEASURES OF VISIBILITY ON CURVES IN THE PLANE FOR POISSON SHADOWING PROCESSES

By

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Abstract

The present study generalizes the methodology developed in [10] to determine visibility probabilities of points in the plane, when shadows are cast by a Poisson field of random objects (disks), given a source of light in the origin. Furthermore, a technique is developed for the determination of the moments of a visibility measure on star-shaped curves in the plane. A numerical example is provided to illustrate the method in some special case.

Key Words: *Visibility Measure, Geometrical Probabilities, Poisson Random Field, Shadowing Process*

1. Introduction

In a previous paper [10] we studied the distribution of a measure of visibility on a circle, or on specified arcs on the circumference of the circle, when random number of disks of random size are randomly distributed within the circle, according to a specified Poisson field. These disks obscure the visibility (cast shadows) on portions of the specified arcs, for an observer (source of light) located at the center. We provided a method of computing the moments of the measure of visibility, to any desired order by a recursive technique. We also approximated the distribution of the normalized measure of visibility on (0,1) by a beta distribution mixed with a two-point discrete distribution. The present study provides a generalization of the previous methodology to any star-shaped curves in the plane, when the random disks are distributed in a specified region according to a Poisson field. This generalization can treat a wide class of geometrical probability problems of line of sight, visibility measures on curves, etc. More specifically, in Section 2 we introduce the general structure of the field of obstacles (disks) and the associated visibility problems. In Section 3 we further develop the basic results to obtain visibility probabilities of points on star-shaped curves in the plane. Section 4 is devoted to the definition and derivation of special functions (K-functions) required for the determination of visibility probabilities. In Section 5 we discuss the measure of visibility on smooth star-shaped curves. We develop, furthermore, recursive formulae for the computation of the moments of the measure of visibility. Section 6 provides a complete theoretical development and numerical computations of a special case of annular region, chosen to illustrate the methodology and show an example of actual computations. It is shown that the mixed-beta distribution provides an excellent approximation to the exact distribution of the measure of visibility, in the special case examined in Section 6.

The literature on shadowing processes is quite limited. Chernoff and Daly [2] studies the distribution of length of shadows of disks on a line. The shadowing problem, however, is a special case of the general coverage problem on which there is extensive literature. In particular

refer to the studies of Robbins [5,6], Ailam [1], Greenberg [4], Siegel [7,8] and the monograph of Solomon [9] which summarizes many of the important results.

2. Poisson Random Fields and Visibility Probabilities

Consider a countable set of disks, $\mathcal{D}$, scattered on the plane. Each disk is specified by a vector (ρ,θ,y) , where ρ , $0 \leq \rho < \infty$, and θ , $-\pi \leq \theta < \pi$ are the polar coordinates of its center with respect to an origin 0 (observation point, source of light); and y , $0 \leq y < y^*$, is its diameter. Let S_0 denote the collection of Borel subsets of the space.

$$S_0 = \{(\rho,\theta,y):\ 0{\leq}\rho{<}\infty,\ -\pi{\leq}\theta{<}\pi,\ 0{\leq}y{<}y^*\}\ . \tag{2.1}$$

For every $B \in S_0$, let $N\{B\}$ denote the number of disks such that $(\rho,\theta,y){\in}B$ (satisfying condition B). If $N\{B\}$ is a random variable for every $B \in S_0$ then the elements of $\mathcal{D}$ are called random disks. In particular, the elements of $\mathcal{D}$ are called Poisson random disks if for every $B \in S_0$, $N\{B\}$ has a Poisson distribution with mean

$$\nu\{B\} = \mu \int \int_B\!\!\int dG(y|\rho,\theta)H(d\rho,d\theta)\ , \tag{2.2}$$

where μ , $0{<}\mu{<}\infty$, is an intensity parameter; $H(\rho,\theta)$ and $G(y|\rho,\theta)$ are, respectively, the sigma-finite measure of the location coordinates (ρ,θ) and the conditional c.d.f. of the diameter, y , given (ρ,θ) .

The special case in which $H(d\rho,d\theta) = \rho d\rho d\theta$ and $G(y|\rho,\theta) = G(y)$, is called the standard case. The standard case is the one in which the centers of the disks are uniformly distributed on the plane and their diameters are independent of their location. In this case

$$\nu\{B\} = \mu\!\int_0^{y^*} H_B(y)dG(y)\ , \tag{2.3}$$

where μ is the mean number of disks per unit area; $H_B(y)$ is the area of the region in which disks of diameter y , satisfying condition B , are centered.

The theoretical framework is not restricted to random disks in the plane, but can be generalized to Poisson random objects in the plane, or even more generally in the space, by replacing the parametric vectors (ρ,θ,y) with a general parametric vectors $(n_1,\ldots,n_p)$, which characterize the location, orientation, shape and size of a family of objects.

In such cases S_0 and $\mathcal{S}_0$ are properly generalized. In order to simplify the presentation, we restrict attention in the present study to the family of random disks in the plane. The approach, however, is valid for general families.

A natural requirement for shadowing processes is that the source of light (the origin) is uncovered. We therefore introduce the <u>structural condition</u>

$$C_0 = \{(\rho,\theta,y);\ \tfrac{y}{2}\leq\rho<\infty,\ -\pi\leq\theta<\pi,\ 0<y<y*\} \tag{2.4}$$

and assume that all random disks satisfy C_0. In special cases one considers more stringent structural conditions, specified by sets C contained in C_0. For example, in the previous paper of Yadin and Zacks [10] the structural condition considered is

$$C = \{(\rho,\theta,y);\ \tfrac{y}{2}\leq\rho<1-\tfrac{y}{2};\ -\pi\leq\theta<\pi,\ 0<y\leq1\}\ .$$

Let P be a point in the plane. P is said to be <u>visible</u> (in light) if the line segment $\overline{OP}$ does not intersect any random disk. The set of all visible points in direction s, $-\pi\leq s<\pi$, starting at the origin, is called a <u>line of sight</u>, L_s. Let $P = (r,s)$ be a point in the plane, in orientation s and distance r from the origin. A random disk (ρ,θ,y) intersects the line segment $\overline{OP}$ if, and only if, (ρ,θ,y) belongs to

$$B(r,s) = \{(\rho,\theta,y);\ (\rho,\theta) \in B(r,s,y),\ 0<y\leq y*\}\ , \tag{2.5}$$

where the set $B(r,s,y)$ is the set of all points having distances from $\overline{0,P}$ smaller than $y/2$. (See Fig. 1.) A disk which intersects a line segment $\overline{OP}$ is said to cast <u>shadow</u> on P. Accordingly, a point $P = (r,s)$ is visible if $N\{B(r,s)\cap C\} = 0$. Hence, a line of sight L_s has magnitude

$$||L_s|| = \sup\{r;\ N\{B(r,s)\cap C\} = 0\}\ . \tag{2.6}$$

Notice that $||L_s||$ is a random variable.

Let $P = (r,s)$ be a point in the plane. Under the Poisson randomness assumption, the probability that P is visible is

$$Q(r,s) = P[N\{B(r,s)\cap C\} = 0]$$
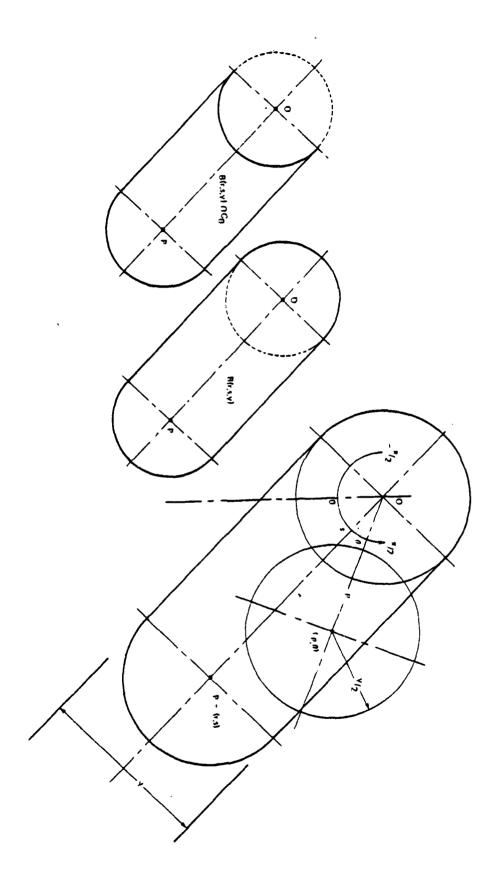$$= \exp\{-\nu\{B(r,s)\cap C\}\}\ , \tag{2.7}$$

Fig. 1. The Geometry Of Disks Casting Shadows On A Point

where $\nu\{B(r,s)\cap C\}$ is obtained according to (2.2). From this function one can immediately obtain the distribution of $||L_s||$. Indeed,

$$P\{||L_s||>\ell\} = Q(\ell,s), \quad 0\leq\ell<\infty \quad . \tag{2.8}$$

Notice that in the standard case

$$\nu\{B(r,s)\cap C_0\} = \mu\xi r \quad , \tag{2.9}$$

where ξ is the expected value of a random diameter, Y , whose distribution is $G(y)$. Indeed, for $Y = y$, the area of $B_0 = B(r,s)\cap C_0$ is $H_{B_0}(y) = ry$, as can be implied from Fig. 1. Accordingly, the distribution of $||L_s||$ in the standard case is exponential with mean $1/\rho$, where $\rho = \mu\xi$. The present result can be further generalized when the random objects are not necessarily disks in the plane. The same type of result for infinitesimal objects is given in Feller [3, p.10].

Let P_1 and P_2 be two points in the plane having coordinates $(r(s_1),s_1)$ and $(r(s_2),s_2)$. A necessary and sufficient condition for P_1 and P_2 to be simultaneously visible is that

$$N\{(B(r(s_1),s_1)\cup B(r(s_2))\cap C\} = 0 \quad .$$

Since $B(r(s_1),s_1)$ and $B(r(s_2),s_2)$ are generally not disjoint it is more convenient to consider a union of disjoint sets obtained in the following manner. Assume that the points (s,θ,y) in C have orientation coordinates in $[s',s'']$. Define, for any s in $[s',s'']$,

$$B_+(s) = \{(\rho,\theta,y): \frac{y}{2\sin(\theta-s)} <\rho<\infty; \; s<\theta<s''; \; 0<y<y*\} \tag{2.10}$$

and

$$B_-(s) = \{(\rho,\theta,y): \frac{y}{2\sin(s-\theta)} <\rho<\infty; \; s'<\theta<s, \; 0<y<y*\} \quad . \tag{2.11}$$

$B_+(s)$ and $B_-(s)$ are sets of all points in C , on the right and on the left of the ray with orientation s , which do not intersect it.

Accordingly, if $s_1<s_2$, $(B(r(s_1),s_1)\cup B(r(s_2),s_2))\cap C = C - (((B_-(s_1)\cup B_+(s_2))\cap C)\cup((B_+(s_1)\cap B_-(s_2))\cap C))$, where $A-B$ is the complement of B with respect to A . Thus, the probability that the points P_1 and P_2 are simultaneously visible is

$$P(s_1,s_2) = \exp\{-[\nu\{C\} - \nu\{(B_-(s_1)\cup B_+(s_2))\cap C\} - \nu\{(B_+(s_1)\cap B_-(s_2))\cap C\}]\} \quad . \tag{2.12}$$

Generally, if $s' \leq s_1 < \ldots < s_n \leq s''$ are the orientation coordinates of n points then the probability that they are simultaneously visible is

$$P(s_1, \ldots, s_n) = \exp\left\{ -\left[\nu\{C\} - \nu\{(B_-(s_1) \cup B_+(s_n)) \cap C\} \right.\right.$$

$$\left.\left. - \sum_{i=1}^{n-1} \nu\{(B_+(s_i) \cap B_-(s_{i+1})) \cap C\} \right]\right\} \quad . \tag{2.13}$$

3. Visibility Probabilities Of Points On Star-Shaped Curves Under A Special Field Structure

Consider a star-shaped curve in the plane, C , such that each ray originating at the origin, O , intersects C at most once. The curve C is specified by a continuous positive function, $r(s)$, on the domain $[s',s'']$, where $-\frac{\pi}{2} \le s' < s'' \le \pi/2$. We furthermore require that $r(s)$ will have almost always continuous derivative.

The field structure is specified by the following assumptions on the random nature of the distributions of the disks. First, we assume that the random diameters, $Y_1, Y_2, \ldots$ of the disks are distributed over the interval $[a,b]$, where $0 \le a < b < \infty$. Furthermore, we specify two star-shaped curves, U and W , between the origin, O , and C , so that the centers of the disks are distributed between U and W , and no disk can either cover the origin or intersect C . More specifically, let

$$U = \{u(\theta); \theta' \le \theta \le \theta''\} \quad \text{and} \quad W = \{w(\theta); \theta' \le \theta \le \theta''\} \quad ,$$

where $[s',s''] \subset [\theta',\theta''] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$. We impose the following field structural conditions (see Fig. 2): Every point (ρ,θ,y) in C_1 , with $\theta \in [\theta',\theta'']$ and $y \in [a,b]$ should satisfy the two conditions

(C.1) $\quad \frac{b}{2} \le u(\theta) < \rho \le w(\theta)$

(C.2) $\quad$ If a point $P = (\xi(s),s)$ is on the circumference of the disk $|(\xi(s),s) - (w(\theta),\theta)| = b/2$, then $w(\theta) \cos (\theta-s) + [(\frac{b}{2})^2 - w^2(\theta) \sin^2(\theta-s)]^{1/2} < r(s)$.

Condition (C.1) guarantees that the origin cannot be covered. Condition (C.2) assures that C is not intersected by any random disk in the field. Since $-\frac{\pi}{2} \le s' < s'' \le \pi/2$, $B_-(s') \cap C_1$ and $B_+(s'') \cap C_1$ are disjoint. Let C be the subset of C_1 containing only disks which may cast shadows on C , i.e., they intersect line segments $\overline{OP}$, where $P \in C$. According to (2.7), (2.10) and (2.11) , the visibility probability of a point $P = (r(s),s)$ on $\quad$ is

$$P\left\{N\{B(r(s),s) \cap C\} = 0\right\} = \tag{3.1}$$

$$\exp\left\{-\left[\nu\{C\} - \nu\{B_-(s) \cap C\} - \nu\{B_+(s) \cap C\}\right]\right\} \quad ,$$

where

$$\nu\{C\} = \nu\{C_1\} - \nu\{B_-(s')\cap C\} - \nu\{B_+(s'')\cap C_1\} \qquad (3.2)$$

and

$$\nu\{C_1\} = \mu \int_{\theta'}^{\theta''} \int_{u(\theta)}^{w(\theta)} H(d\xi, d\theta) \quad . \qquad (3.3)$$

Simultaneous visibility probabilities of several points on C are given generally by formula (2.13). We develop now the technique of determining the expected number of points in the sets $B_-(s)\cap C$ or $B_+(s)\cap C$.

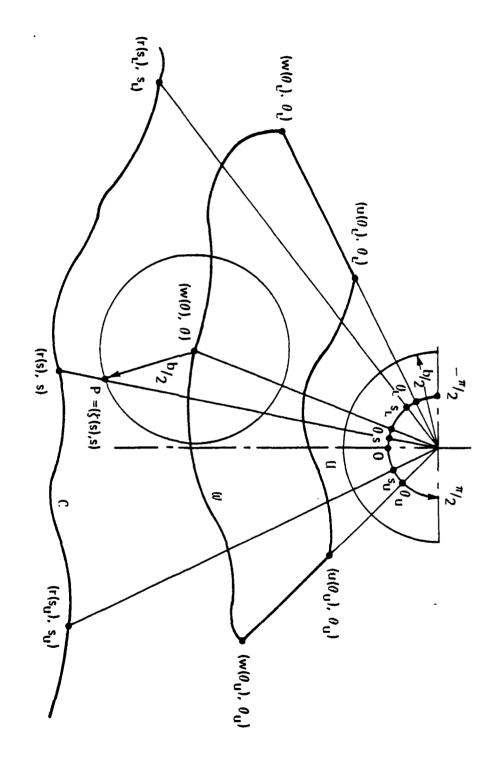Fig. 2. The Geometry Of Disks Casting Shadows On A Curve

4. Definition And Derivation of the K-Functions

Given a point $P = (r(s), s)$ on C, let $\mu K_-(s,t)$ and $\mu K_+(s,t)$ denote the expected numbers of disks $(\rho, \theta, y) \in C_1$ with $s-t \leq \theta \leq s$ and $s \leq \theta \leq s+t$, respectively, which do not intersect the line segment $\overline{OP}$. These functions are given by

$$K_-(s,t) = \int_{s-t}^{s} \int_{u(\theta)}^{w(\theta)} G(y(\rho, s-\theta) | \rho, \theta) H(d\rho, d\theta) \quad ,$$

and   (4.1)

$$K_+(s,t) = \int_{s}^{s+t} \int_{u(\theta)}^{w(\theta)} G(y(\rho, \theta-s) | \rho, \theta) H(d\rho, d\theta) \quad .$$

where

$$y(\rho, \tau) = \begin{cases} 2\rho \sin\tau & , & \text{if } \tau < \pi/2 \\ 2\rho & , & \text{if } \tau \geq \pi/2 \end{cases} \quad (4.2)$$

Indeed, when $|\theta - s| < \pi/2$, a disk centered at (ρ, θ) does not intersect $\overline{OP}$ if its diameter is smaller than $2\rho \sin|\theta - s|$. If, however, $|\theta - s| \geq \pi/2$, the diameter of the disk should not exceed 2ρ.

Thus, according to (3.2) and (4.1),

$$\nu\{C\} = \nu\{C_1\} - \mu[K_-(s', s'-\theta') + K_+(s'', \theta''-s'')] \quad . \quad (4.3)$$

Similarly, for all $s \in [s', s'']$,

$$\nu\{B_-(s) \cap C\} = \mu[K_-(s, s-\theta') - K_-(s', s'-\theta')]$$

(4.4)

$$\nu\{\overline{B}_+(s) \cap C\} = \mu[K_+(s, \theta''-s) - K_+(s'', \theta''-s'')] \quad .$$

Furthermore, for $s' \leq s_1 < s_2 \leq s''$,

$$\nu\{B_+(s_1) \cap B_-(s_2) \cap C\} =$$

$$\mu[K_+(s_1, \frac{s_2-s_1}{2}) + K_-(s_2, \frac{s_2-s_1}{2})] \quad . \quad (4.5)$$

The value $t = \frac{s_2-s_1}{2}$ was selected in (4.5) to avoid the possibility that a disk centered at (ρ, θ) with $s < \theta < s+t$, which does not intersect $\overline{OP_1}$ will nevertheless intersect $\overline{OP_2}$, and vice versa.

For the purpose of evaluating (4.1) we introduce the auxiliary functions

$$K_-(s,d\theta,\nu) = \int_0^\nu G(y(\rho,s-\theta)|\rho,\theta)H(d\rho,d\theta)$$

and

$$(4.6)$$

$$K_+(s,d\theta,\nu) = \int_0^\nu G(y(\rho,\theta-s)|\rho,\theta)H(d\rho,d\theta)$$

The K-functions are thus given by

$$K_-(s,t) = \int_{s-t}^s [K_-(s,d\theta,w(\theta)) - K_-(s,d\theta,u(\theta))]$$

and

$$(4.7)$$

$$K_+(s,t) = \int_s^{s+t} [K_+(s,d\theta,w(\theta)) - K_+(s,d\theta,u(\theta))] \ .$$

Notice that $G(y|\rho,\theta) = 0$ for all $y<a$ and $G(y|\rho,\theta) = 1$ for all $y\geq b$. Accordingly, from (4.6)

$$K_+(s,d\theta,\nu) = \int_{A(\theta-s,\nu,a)}^{A(\theta-s,\nu,b)} G(y(\rho,\theta-s)|\rho,\theta)H(d\rho,d\theta) \ +$$

$$\int_{A(\theta-s,\nu,a)}^\nu H(d\rho,d\theta) \qquad , \qquad (4.8)$$

where

$$A(\tau,\nu,x) = \begin{cases} \min(\nu,\dfrac{x}{2\sin\tau}) & , \quad \text{if} \quad \tau<\pi/2 \\[2mm] \min(\nu,\dfrac{x}{2}) & , \quad \text{if} \quad \tau\geq\pi/2 \ . \end{cases}$$

$$(4.9)$$

The function $K_-(s,d\theta,\nu)$ can be obtained from (4.8) by replacing $\theta-s$ by $s-\theta$. In the standard case we substitute in the above formula $H(d\rho,d\theta) = \rho d\rho d\theta$ and $G(y|\rho,\theta) = G(y)$. As an example, we will continue the derivations of the standard case and a uniform distribution of Y over [a,b] , i.e.,

$$G(y) = \begin{cases} 0 & , \quad y<a \\[2mm] \dfrac{y-a}{b-a} & , \quad a\leq y\leq b \\[2mm] 1 & , \quad y>b \end{cases} \qquad (4.10)$$

Under these assumptions one obtains, for $\theta > s$,

$$K_+(s,d\theta,\nu) = \frac{d\theta}{b-a} \int\limits_{A(\theta-s,\nu,a)}^{A(\theta-s,\nu,b)} (y(\rho,\theta-s)-a)\rho\,d\rho +$$

$$+ \frac{d\theta}{2}(\nu^2 - A^2(\theta-s,\nu,b)) =$$

$$= \begin{cases} 0 & \text{, if } \theta-s < \sin^{-1}(\frac{a}{2\nu}) \\[2ex] \dfrac{d\theta}{b-a}[\dfrac{2}{3}\nu^3\sin(\theta-s) - \dfrac{a}{2}\nu^2 + \dfrac{1}{24}\dfrac{a^3}{\sin^2(\theta-s)}] & \text{, if } \sin^{-1}(\frac{a}{2\nu}) \leq \theta-s < \sin^{-1}(\frac{b}{2\nu}) \\[2ex] \dfrac{d\theta}{2}[\nu^2 - \dfrac{a^2+ab+b^2}{12\sin^2(\theta-s)}] & \text{, if } \sin^{-1}(\frac{b}{2\nu}) \leq \theta-s < \pi/2 \\[2ex] \dfrac{d\theta}{2}[\nu^2 - \dfrac{a^2+ab+b^2}{12}] & \text{, if } \theta-s \geq \pi/2 \;. \end{cases} \qquad (4.11)$$

As seen in (4.11), the function $K_+(s,d\theta,\nu)$, in the standard-uniform case, be written as $K_+(s,d\theta,\nu) = \Lambda(\theta-s,\nu,a,b)d\theta$. Similarly, $K_-(s,d\theta,\nu) = \Lambda(s-\theta,\nu,a,b)d\theta$, where $\Lambda(\theta-s,\nu,a,b)$ designates the R.H.S. of (4.11).

5. The Measure of Visibility on C and Its Moments

Define the indicator function $I(s)$, $s' \leq s \leq s''$ so that $I(s) = 1$ if $(r(s), s)$ is a visible point and $I(s) = 0$ otherwise. The total measure of the visible part of C is given by

$$V\{C\} = \int_{s'}^{s''} I(s)[r^2(s) + (r'(s))^2]^{1/2} ds \quad . \tag{5.1}$$

$V\{C\}$ is the __visibility measure__ of C. In coverage problems it is known also as the measure of vacancy (Ailam[1]).

The first moment (expected value) of $V(C)$ is

$$E\{V(C)\} = \int_{s'}^{s''} E\{I(s)\}\ell(s)ds \quad , \tag{5.2}$$

where $\ell(s) = [r^2(s) + (r'(s))^2]^{1/2}$. Moreover,

$$E\{I(s)\} = Q(r(s), s)$$

$$= \exp\{-\nu\{C_1\} + \mu[K_-(s, s-\theta') +$$

$$K_+(s, \theta''-s)]\} \quad . \tag{5.3}$$

$V\{C_1\}$ is given by (3.3) In the standard case it assumes the form

$$V\{C_1\} = \frac{\mu}{2} \int_{\theta'}^{\theta''} [w^2(\theta) - u^2(\theta)]d\theta \quad . \tag{5.4}$$

Generally, for every $n \geq 2$, the n-th moment of $V\{C\}$ is given by

$$E\{V^n\{C\}\} = \int_{s'}^{s''} \cdots \int_{s'}^{s''} E\{\prod_{i=1}^{n} I(s_i)\} \prod_{i=1}^{n} \ell(s_i)ds_i$$

$$= n! \int \cdots \int_{s \leq s_1 \leq \ldots \leq s_n \leq s''} P(s_1, \ldots, s_n) \prod_{i=1}^{n} \ell(s_i)ds_i \tag{5.5}$$

Indeed, $E\{\prod_{i=1}^{n} I(s_i)\}$ is the probability that all the n points are simultaneously visible. According to (2.13), (4.3), (4.4), and (4.5) we

obtain, for every $s' \leq s_1 \leq \ldots \leq s_n \leq s''$,

$$P(s_1,\ldots,s_n) = \exp\{-\nu\{C_1\}\}\exp\{\mu K_-(s_1,s_1-\theta')$$

$$+ \mu K_+(s_n,\theta''-s_n) + \mu \sum_{i=1}^{n-1}[K_+(s_i, \frac{s_{i+1}-s_i}{2}) + K_-(s_{i+1}, \frac{s_{i+1}-s_i}{2})]\} \qquad (5.6)$$

The n-th moment of $V\{C\}$, for $n \geq 1$, can be computed according to (5.5) and (5.6) in the following recursive manner. Define first

$$\Psi_0(s) = \exp\{\mu K_-(s,s-\theta')\} \qquad (5.7)$$

and for $j=1,2,\ldots,n-1$ define

$$\Psi_j(s) = \int_{s'}^{s} \ell(y)\Psi_{j-1}(y)\exp\{\mu[K_+(y, \frac{s-y}{2}) +$$

$$K_-(s, \frac{s-y}{2})]\}dy \qquad . \qquad (5.8)$$

Then, for each $n \geq 1$,

$$E\{V^n\{C\}\} = n! \exp\{-\nu\{C_1\}\} \cdot$$

$$\cdot \int_{s'}^{s''} \ell(y)\Psi_{n-1}(y)\exp\{\mu K_+(y,\theta'' -y)\}dy \qquad . \qquad (5.9)$$

Finally, let $L\{C\}$ denote the length of C . The distribution of $V\{C\}/L\{C\}$ is concentrated on $[0,1]$, with jumps at the two end points 0 and 1 and absolutely continuous elsewhere. It follows that

$$\lim_{n\to\infty} E\{V^n\{C\}\}/L^n\{C\} = P\{V\{C\} = L\{C\}\} \qquad . \qquad (5.10)$$

6. Numerical Determination of the Moments of Visibility in the Standard-Uniform Case and Annular Sector of Disks

In the present section we continue the development of the formulae for the recursive determination of the moments, when the disks are distributed over the annular sector with

$$r(s) = r \qquad , \quad -\frac{\pi}{2} < s' \leq s \leq s'' < \frac{\pi}{2}$$

$$w(\theta) = w \qquad , \quad -\frac{\pi}{2} \leq \theta' \leq \theta \leq \theta'' \leq \frac{\pi}{2}$$

and

$$u(\theta) = u \qquad , \quad -\frac{\pi}{2} \leq \theta' \leq \theta \leq \theta'' \leq \frac{\pi}{2} \qquad .$$

Here, $0 < b/2 < u < w < r - b/2$.

We consider the standard case, where the distribution of the disk diameters, Y , is uniform (4.10). Thus, according to (4.7) and (4.11), the $K(s,t)$ functions, in the present case, do not depend on s , and satisfy

$$K_+(s,t) = K_-(s,t) = K^*(t,w) - K^*(t,u) \quad , \tag{6.1}$$

for all s , $s' \leq s \leq s''$, where

$$K^*(t,v) = \int_0^t \Lambda(\tau,v,a,b)d\tau$$

$$= \begin{cases} 0 & , \text{ if } t \leq \sin^{-1}(\frac{a}{2v}) \\ \\ K_1(t,v) & , \text{ if } \sin^{-1}(\frac{a}{2v}) \leq t < \sin^{-1}(\frac{b}{2v}) \\ \\ K_2(t,v) & , \text{ if } \sin^{-1}(\frac{a}{2v}) \leq t < \pi/2 \\ \\ K_3(t,v) & , \text{ if } t \geq \pi/2 \end{cases} \tag{6.2}$$

in which

$$K_1(t,v) = \frac{1}{b-a}\left\{\frac{v^2}{3}\left[(4v^2-a^2)^{1/2} - 2v\cos t\right] - \frac{a}{2}v^2\left[t - \sin^{-1}(\frac{a}{2v})\right] + \frac{a^2}{24}\left[(4v^2-a^2)^{1/2} - a\cot(t)\right]\right\} \quad , \tag{6.3}$$

$$K_2(t,v) = \frac{1}{b-a}\left\{\frac{v^2}{3}\left[(4v^2-a^2)^{1/2} - (4v^2-b^2)^{1/2}\right] - \frac{a}{2}v^2\left[\sin^{-1}(\frac{b}{2v})\right.\right.$$

$$\left.- \sin^{-1}(\frac{a}{2v})\right] + \frac{a^3}{24}\left[\frac{1}{a}(4v^2-a^2)^{1/2} - \frac{1}{b}(4v^2-b^2)^{1/2}\right]\right\}$$

$$+ \frac{v^2}{2}(t - \sin^{-1}(\frac{b}{2v})) - \frac{a^2+ab+b^2}{24}(\frac{1}{b}(4v^2-b^2)^{1/2}$$

$$- \text{cotan } t) \quad , \tag{6.4}$$

and

$$K_3(t,v) = K_2(\frac{\pi}{2},v) + (t - \frac{\pi}{2})\left[\frac{v^2}{2} - \frac{a^2+ab+b^2}{24}\right] \quad . \tag{6.5}$$

Notice that in the present case, $\ell(s) = r$ for all $s' \leq s \leq s''$. Hence, the moments of $V\{C\}$ can be determined according to (5.4), (5.7)-(5.9) and (6.1) in the following manner.

Let

$$\lambda = \exp\{-\frac{\mu}{2}(w^2-u^2)(\theta''-\theta')\} \quad , \tag{6.6}$$

$$\Psi_0(s) = \exp\{\mu[K^*(s-\theta',w) - K^*(s-\theta',u)]\} \quad , \tag{6.7}$$

and

$$H(s) = \exp\{\mu[K^*(\theta''-s,w) - K^*(\theta''-s,u)]\} \tag{6.8}$$

for $s' \leq s \leq s''$.

Define recursively, for every $j \geq 1$,

$$\Psi_j(s) = \int_{s'}^{s} \Psi_{j-1}(y)\exp\{2\mu[K^*(\frac{s-y}{2},w) - K^*(\frac{s-y}{2},u)]\}dy \quad . \tag{6.9}$$

The n-th moment of $V\{C\}$ is then

$$\mu_n = \lambda n! r^n \int_{s'}^{s''} \Psi_{n-1}(s)H(s)ds \quad . \tag{6.10}$$

Let P_1 be the probability that $V\{C\} = L\{C\}$, which is the probability that C is completely visible. According to (3.2) and (3.3)

$$P_1 = \lambda \exp\{\mu[K^*(s'-\theta',w) - K^*(s'-\theta',u) +$$

$$K^*(\theta''-s'',w) - K^*(\theta''-s'',u)]\} \tag{6.11}$$

The numerical determination of the moments μ_n was performed in the following manner. The interval $[s',s'']$ was partitioned to M sub-intervals of length $\Delta = (s''-s')/M$. The grid points for the computation are then $s_j = s'+j\Delta$, $j=0,1,\ldots,M$. The functions $\Psi_o(s)$ and $G(s)$ were computed exactly for each $j=0,\ldots,M$. The functions $\Psi_k(s), k\geq 1$, were then computed at the grid points s_j, according to the numerical integration formula

$$\Psi_k(s_j) = \begin{cases} \overline{0} & \text{, if } j=0 \\[2mm] \frac{\Delta}{2}(\Psi_{k-1}(s_o)\exp\{2\mu\,\widetilde{K}\,(\frac{\Delta}{2})\} + \Psi_{k-1}(s_1)) & \text{, if } j=1 \\[2mm] \frac{\Delta}{2}(\Psi_{k-1}(s_o)\exp\{2\mu\,\widetilde{K}\,(\frac{j\Delta}{2})\} + \Psi_{k-1}(s_1)) + \\[1mm] \quad + \Delta \sum_{i=1}^{j-1}\Psi_{k-1}(s_i)\exp\{2\mu\,\widetilde{K}\,(\frac{(j-i)\Delta}{2})\} & \text{, if } j\geq 2 \end{cases}$$ (6.12)

where $\widetilde{K}(t) = K*(t,w) - K*(t,u)$. Finally, the n-th moment of $V(C)$ is determined by the formula

$$\mu_n^* = \lambda n! r^n \Delta\{(\Psi_{n-1}(s_o)H(s_o) + \Psi_{n-1}(s_m)H(s_m))/2$$
$$+ \sum_{j=1}^{m-1}\Psi_{n-1}(s_j)H(s_j)\}\quad.$$ (6.13)

If the value of M is large one obtains very good approximation by applying formulae (6.12)-(6.13). In Table 6.1 we provide the first ten normalized moments of $V\{C\}$, i.e., $\mu_n^*/(r(s''-s'))^n$, for the case of $s' = -\pi/18$, $s'' = -s'$, $M=60$, $r=1$, $u=.5$, $w=.75$, $a=.1$ and $b=.3$. The values of μ (the Poisson intensity) are $1(2)9$.

Table 6.1 The Normalized Moments of V(C) and Their Beta-Mixture
          Approximations, for s'=π/18, s''=-s', M=60, r=1., u=.5,
          w=.75, a=.1, b=.3

Intensity	Normalized Moments										
μ	1	2	3	4	5	6	7	8	9	10	∞
1.	0.951	0.934	0.926	0.921	0.918	0.917	0.917	0.919	0.921	0.925	.901
	0.951	0.934	0.926	0.921	0.917	0.915	0.913	0.912	0.910	0.910	
3.	0.860	0.816	0.794	0.781	0.773	0.768	0.765	0.763	0.763	0.765	.730
	0.860	0.816	0.794	0.780	0.772	0.765	0.761	0.757	0.754	0.752	
5.	0.778	0.713	0.681	0.662	0.650	0.643	0.638	0.634	0.633	0.633	.592
	0.778	0.713	0.681	0.662	0.649	0.640	0.634	0.629	0.625	0.621	
7.	0.703	0.623	0.584	0.562	0.548	0.538	0.532	0.527	0.525	0.524	.480
	0.703	0.623	0.584	0.562	0.547	0.536	0.528	0.522	0.518	0.514	
9.	0.636	0.545	0.502	0.477	0.461	0.451	0.444	0.439	0.435	0.434	.389
	0.636	0.545	0.502	0.477	0.460	0.449	0.440	0.434	0.429	0.425	

Notice that $\mu_\infty^* = P_1$.

As in the previous study of Yadin and Zacks [10], the distribution of
$V(C)/r(s''-s')$ is approximated by a mixture of a beta-distribution with
a discrete distribution concentrated on 0 and 1 . The mixed distri-
bution of the normalized visibility measure, $v^*(C) = V(C)/r(s''-s')$,
has a c.d.f.

$$F^*(x) = \begin{cases} 0 & \text{, if } x<0 \\ P_0 + (1-P_0-P_1) \frac{1}{B(\alpha,\beta)} \int_0^x y^{\alpha-1}(1-y)^{\beta-1}dy & \text{, if } 0 \leq x < 1 \\ 1 & \text{, if } x \geq 1 \end{cases} \tag{6.14}$$

Since the value of P_1 is known, we determine $f^*(x)$, in the various
cases, by equating the first three moments of (6.14) to the moments of
$v^*(C)$. The n-th moment of (6.14) is

$$\tilde{\mu}_n = P_1 + (1-P_0-P_1) \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{(\alpha+\beta)(\alpha+\beta+1)\ldots(\alpha+\beta+n-1)} , \quad n \geq 1 \tag{6.15}$$

Accordingly, if $c_n = \mu_n^* - P_1$, then

$$\alpha = \frac{2c_2^2 - {}_3(c_1 + c_2)}{c_1 c_3 - c_2^2} \quad ,$$

$$\beta = \frac{(c_1 - c_2)(c_2 - c_3)}{c_1 c_3 - c_2^2} \tag{6.16}$$

and

$$P_0 = 1 - P_1 - \frac{c_1(\alpha + \beta)}{\alpha} \quad .$$

In Table 6.2 we provide the values of $\sigma = (\mu_2^* - (\mu_1^*)^2)^{1/2}$, P_0, P_1, α and β corresponding to the cases of Table 6.1. We have also computed the first ten moments of (6.14) according to the parameters determined by the true moments, μ_n^* , and presented them in Table 6.1 (lower line of each case).

We see that the actual moments and the ones of the approximating distribution are extremely close. This indicates that the mixed-beta distribution is apparently very close to the true one. We obtain in this manner also an estimate of the probability of complete coverage, P_0 . An explicit formula for this parameter is not yet available.

Table 6.2 <u>Parameters of the Mixed-Beta Approximating Distribution</u>
(Specifying parameters as in Table 6.1)

Intensity μ	σ	P_0	P_1	α	β
1	.1733	.0020	.9005	1.1161	1.0398
3	.2765	.0152	.7302	1.1405	1.0967
5	.3290	.0326	.5921	1.0956	1.1194
7	.3590	.0536	.4801	1.0450	1.1386
9	.3754	.0768	.3893	0.9937	1.0967

In order to realize the effect of different case parameters on the results, we present in Table 6.3 the first 10 moments, μ_n^*, of $V^*(C)$, for a longer curve $(s'=-\pi/3, s''=-s')$. The computation was performed on a grid of M=180 subintervals. The intensities μ are 1(2)9, and the distribution of the disk diameters is uniform on (0.1,0.3).

We do not present the moments of (6.14), which correspond to these cases, since they are very close to the actual moments. The results of the computations presented in Table 6.3 show similar trends to those indicated in Tables 6.1 and 6.2.

Table 6.3 <u>Normalized Moments</u>, $V^*(C)$, <u>And The Parameters of The Mixed Beta Distribution for the Cases Of</u> s'=-π/3, s''=-s', r=1., μ=.5, W=.75, a=.1, b=.3 <u>and</u> M=180.

μ	1	2	3	4	5	6	7	8	9	10	∞
1	.951	.912	.880	.854	.832	.814	.799	.787	.776	.767	.686
3	.860	.758	.682	.623	.577	.541	.511	.487	.467	.451	.322
5	.778	.631	.529	.456	.402	.360	.328	.303	.282	.266	.152
7	.703	.526	.412	.335	.280	.241	.212	.189	.171	.157	.071
9	.636	.438	.321	.246	.196	.162	.137	.118	.104	.094	.033

Table 6.5 <u>Parameters of The Mixed-Beta Approximating Distribution (Specifying Parameters as in Table 6.3)</u>

μ	σ	P_0	P_1	α	β
1	.0856	0	.6856	15.8793	2.9360
3	.1364	0	.3223	8.9669	2.3286
5	.1619	0	.1515	6.3166	2.2383
7	.1762	0	.0712	4.9572	2.3287
9	.1838	0	.0335	4.1394	2.5105

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The present study generalizes the methodology developed in [10] to determine visibility probabilities of points in the plane, when shadows are cast by a Poisson field of random objects (disks), given a source of light in the origin. Furthermore, a technique is developed for the determination of the moments of a visibility measure on star-shaped curves in the plane. A numerical example is provided to illustrate the method in some special case.

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